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Quasiclassical approximation for ultralocal scalar fields

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Abstract. We show how to obtain the quasiclassical evolution of a class of field theories called ultralocal fields. Coherent states that follow the 'classical' orbit as defined by Klauder's weak correspondence principle and restricted action principle is explicitly shown to approximate the quantum evolution as $\hbar \to 0$.

1. Introduction

Ultralocal field theory studies a class of models which differ from relativistic theories by the absence of spatial gradients in the Hamiltonian, e.g. the term $(\nabla\phi)^2$ for scalar fields.

Exact operator solutions have already been obtained for these models [1, 2] and they may well provide an alternative route to the perturbative study of quantum field theory where the spatial gradients are to be included as perturbations about the exact, ultralocal solutions. However, up to this date, no one has yet succeeded in finding a representation for the products of spatial derivatives in the context of ultralocal representations.

Ultralocal field theory seems to be particularly appropriate for the study of strong coupling limits to relativistic field theories. For Yang-Mills fields this limit consists of dropping the magnetic field terms from the Hamiltonian, i.e. the terms containing spatial derivatives. Similarly, for general relativity, dropping the spatial scalar Ricci curvature term from the Hamiltonian generator can be considered as a strong coupling limit [3]. Although strong coupling Yang-Mills has been studied in the context of lattice gauge theory, this is not mandatory and application of ultralocal methods seems to be promising. The coordinate invariance of general relativity makes going to a lattice very unnatural and ultralocality ideas may prove to be a sound alternative to understand the quantised theory.

In this paper we address the problem of determining a classical limit of an ultralocal scalar quantum field. The extension of these ideas to general relativity has already been worked out and can be found in [4]. Not only canonical but also affine [2] commutation relations will be discussed. The motivation for this kind of field lies in the fact that in some physical systems the dynamical variables are constrained one way or another. This is the case for example when one wants to restrict the spectrum of the field operator to the positive real line. Then canonical commutation relations are not suitable since the canonical momentum will not be self-adjoint under this restriction. General relativity exhibits this peculiarity since the metric tensor on a

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Cauchy hypersurface must have definite signature. Thus for completeness we treat in this paper both affine and canonical fields.

The paper is organised as follows. Section 1 contains an illustration of how one obtains classical limits in the context of quantum mechanics with the help of coherent states and the role of the weak correspondence principle (WCP) and the restricted action principle (RAP) when one tries to generalise these results to field theory. In § 2 we present a general discussion about ultralocal scalar fields. The representation we choose is specified by a kind of generalised Gaussian vacuum state as this is the most convenient way to construct coherent states. The evolution problem is tackled in § 3 and the coherent states as given by the WCP and RAP will be shown to approximate the evolution of the wavefunction as $\hbar \to 0$, in analogy with the quantum mechanics case. Section 4 contains a discussion and the conclusion.

2. General remarks on classical limits

2.1. Quantum mechanics

We start by reviewing briefly in the context of quantum mechanics the methods on which we will base most of our discussion of taking the classical limit of ultralocal field theory. We work in finite dimensions for simplicity but the extension to two-dimensional boson scalar fields is found in Hepp [5]. Our presentation will be mostly based on this reference (see also Klauder [6] and the book by Thirring [7]).

Consider a canonical system with the real Hamiltonian function

$$\mathscr{H}(p, q) = p^2/2m + V(q) \tag{2.1}$$

in the $2n$-dimensional space $\mathbb{R}^{2n}$ with $(p, q) \in \mathbb{R}^{2n}$. If grad $V \equiv \nabla V$ is Lipschitz around q, then the canonical equations of motion

$$m\dot{q}(t) = p(t) \qquad \dot{p}(t) = -\nabla V(q(t)) \tag{2.2a}$$

always have a unique local solution for $t \in (-a, a)$, $a > 0$, with initial data

$$q \equiv q(0) \qquad p \equiv p(0). \tag{2.2b}$$

The corresponding quantum mechanical problem

$$\mathrm{i}\hbar(\partial/\partial t)\psi(x, t) = \hat{\mathscr{H}}\psi(x, t) \equiv (-\hbar^2\nabla^2/2m + V(x))\psi(x, t) \tag{2.3}$$

in the Hilbert space $H = L^2(\mathbb{R}^n)$ with inner product $(\psi, \phi) = \int \mathrm{d}x\, \psi^* \phi$ always has global solutions if the self-adjoint extensions of ∇^2 and V have a common dense domain D with $\Psi(\cdot, 0) \in D$. This solution

$$\psi(t) = \exp[-(\mathrm{i}/\hbar)\hat{\mathscr{H}}_{\mathrm{sa}} t]\psi(0) \equiv U(t)\psi(0) \tag{2.4}$$

is expressed in terms of any self-adjoint extension $\hat{\mathscr{H}}_{\mathrm{sa}}$ of the operator $-\hbar^2\nabla^2/2m + V$. In the following we will not make any distinction between $\hat{\mathscr{H}}$ and $\hat{\mathscr{H}}_{\mathrm{sa}}$.

The discussion of the connection between (2.2) and (2.3) is as old as quantum mechanics itself [8]. Several methods have been devised to study this problem. The WKB method relates an asymptotic expansion of solutions of (2.3) for $\hbar \to 0$ to solutions of the Hamilton–Jacobi equation [9, 10] for (2.2). (An application of the WKB approximation to the gravitational field can be found in the work of Gerlach [11].) The Feynman integral [12] appears to be a very flexible tool but its use is beyond the scope of this paper. The simplest connection between quantum and classical mechanics, however, goes back to the *Ehrenfest theorem* [13]: for every $\psi \in D$ and V sufficiently regular

$$\begin{aligned}(\mathrm{d}/\mathrm{d}t)(\psi(t), \hat{Q}\psi(t)) &= (1/m)(\psi(t), \hat{P}\psi(t)) \\ (\mathrm{d}/\mathrm{d}t)(\psi(t), \hat{P}\psi(t)) &= -(\psi(t), \nabla V\psi(t))\end{aligned} \tag{2.5}$$

where $\hat{P} = -\mathrm{i}h\nabla$ and $\hat{Q} = x$. However, (2.5) does not define a solution of (2.2) since $(\psi(t), \nabla V\psi(t)) \neq \nabla V(\psi(t), \hat{Q}\psi(t))$ unless V is linear. In general only for $\hbar \to 0$ these expectation values define a solution to (2.2). Another way to relate (2.5) and (2.2) when $\hbar \to 0$ is to use minimal uncertainty states for $\hat{P}$ and $\hat{Q}$, i.e. coherent states [14, 15]. Since coherent states provide the main tool for obtaining the quasiclassical approximation for ultralocal fields in § 3 we give below an idea of how the method works in one-dimensional quantum mechanics.

In order to have the powers of $\hbar$ on the right place it is convenient to use a symmetric representation [5] of the CCR (canonical commutation relations)

$$\hat{p}_\hbar = \sqrt{\hbar}\hat{p} \qquad \hat{q}_\hbar = \sqrt{\hbar}\hat{q} \tag{2.6}$$

where $\hat{p} = -\mathrm{i}\,\mathrm{d}/\mathrm{d}x$ and $\hat{q} = x$. Let $\alpha = (q + \mathrm{i}p)/\sqrt{2} \in \mathbb{C}$ with p, q real parameters and define

$$U(\alpha) = \exp(\alpha\hat{a}^* - \alpha^*\hat{a}) = \exp \mathrm{i}(p\hat{q} - q\hat{p}) \tag{2.7}$$

where $\hat{a} = (\hat{q} + \mathrm{i}\hat{p})/\sqrt{2}$. Using the Campbell-Hausdorff formula $\mathrm{e}^{Y}X\,\mathrm{e}^{-Y} = X + [Y, X] + (1/2!)[Y, [Y, X]] + \dots$ we obtain

$$U(\alpha)^*\hat{a}U(\alpha) = \hat{a} + \alpha. \tag{2.8}$$

Equation (8) implies that in the *coherent state*

$$|\alpha\rangle \equiv U(\alpha)|0\rangle \tag{2.9}$$

with $\hat{a}|0\rangle = 0$; for an arbitrary monomial in the $p_\hbar$ and $q_\hbar$, one has

$$\lim_{\hbar\to 0}\left\langle \frac{1}{\sqrt{\hbar}}\alpha \,\middle|\, \hat{q}_\hbar \dots \hat{p}_\hbar \,\middle|\, \frac{1}{\sqrt{\hbar}}\alpha \right\rangle = q \dots p. \tag{2.10}$$

As an aside notice that in some cases we have to assume that the vacuum state satisfies the factorisation property

$$\lim_{\hbar\to 0}\langle 0|\hat{q}\dots\hat{p}|0\rangle = \langle 0|\hat{q}|0\rangle \dots \langle 0|\hat{p}|0\rangle \tag{2.11}$$

and $\langle 0|\hat{q}|0\rangle = 0 = \langle 0|\hat{p}|0\rangle$ in order for (2.10) to hold (see Yaffe [16, 17]). Hepp [5] shows that (2.10) is preserved under time evolution as given by $U(t)$ in (2.4),

$$\lim_{\hbar\to 0}\left\langle \frac{1}{\sqrt{\hbar}}\alpha \,\middle|\, \hat{q}_\hbar(t) \dots \hat{p}_\hbar(t) \,\middle|\, \frac{1}{\sqrt{\hbar}}\alpha \right\rangle = q(t) \dots p(t) \tag{2.12}$$

as long as the classical orbit $q(t)$, $p(t)$ specified by (2) exists and V is of class C^3 and decreases sufficiently fast when $|x| \to \infty$. Equation (2.12) is the coherent state version of the Ehrenfest theorem in the classical limit.

The fact that along coherent states the quantum mechanical evolution

$$\langle \hbar^{-1/2}\alpha|\hat{a}_\hbar(t)|\hbar^{-1/2}\alpha\rangle$$

and the classical evolution $a(t) = \langle \hbar^{-1/2}\alpha(t)|\hat{a}_\hbar|\hbar^{-1/2}\alpha(t)\rangle$ are in 'weak correspondence', which becomes exact for $\hbar \to 0$, has been analysed by Klauder [6] (here $\hat{a}_\hbar(t) = U(t)^*\hat{a}_\hbar U(t) = (1/\sqrt{2})(\hat{q}_\hbar(t) + \mathrm{i}\hat{p}_\hbar(t))$ and $\alpha(t) = (1/\sqrt{2})(q(t) + \mathrm{i}p(t))$).

However the only rigorous proof of this fact was given by Hepp. Under the same hypotheses underlying (2.12) he also proves the following important formula

$$\lim_{\hbar\to 0}\left\| U(t)U\left(\frac{1}{\sqrt{\hbar}}\alpha\right)|0\rangle - U\left(\frac{1}{\sqrt{\hbar}}\alpha(t)\right)|0\rangle\right\| = 0. \tag{2.13}$$

This equation is constantly mentioned in Klauder's work [18] but never proven. He writes it as

$$\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{\mathscr{H}}t\right)|p_0, q_0\rangle \overset{\hbar\to 0}{\approx} |p_{\mathrm{cl}}(t), q_{\mathrm{cl}}(t)\rangle \tag{2.14}$$

where $|p_{\mathrm{cl}}(t), q_{\mathrm{cl}}(t)\rangle = \exp[-(\mathrm{i}/\hbar)p_{\mathrm{cl}}(t)\hat{Q}]\exp[(\mathrm{i}/\hbar)q_{\mathrm{cl}}(t)\hat{P}]|0\rangle$, $p_0 \equiv p_{\mathrm{cl}}(0)$, $q_0 \equiv q_{\mathrm{cl}}(0)$; the subscript 'cl' refers to the classical orbit.

There are some systems for which (14) is simply

$$\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{\mathscr{H}}t\right)|p_0, q_0\rangle = |p_{\mathrm{cl}}(t), q_{\mathrm{cl}}(t)\rangle. \tag{2.15}$$

In these cases the initial coherent state is not deformed during the evolution but remains a coherent state along the classical orbit. Such systems are called *exact systems.*

2.2. Generalisations to field theory

The action principle leading to the Schrödinger equation (3) is based on the quantum action functional

$$I(\psi) = \int \mathrm{d}t(\psi(t), \mathrm{i}\hbar\dot{\psi}(t) - \hat{\mathscr{H}}\psi(t)) \tag{2.16}$$

where the dot denotes the t derivative. If we replace ψ by a coherent state then the *restricted action*

$$I(p, q) = \int \mathrm{d}t\langle p(t), q(t)|(\mathrm{i}\hbar\partial/\partial t - \hat{\mathscr{H}})|p(t), q(t)\rangle \tag{2.17}$$

leads, upon arbitrary variation of p and q, to the classical equations of motion (2.2a): this is the content of the restricted action principle [18], RAP. To arrive at this result just notice that (2.17) can be re-expressed as (assume $\langle 0|\hat{P}|0\rangle = 0 = \langle 0|\hat{Q}|0\rangle$)

$$I(p, q) = \int \mathrm{d}t[p(t)\dot{q}(t) - \mathscr{H}(p(t), q(t))] \tag{2.18}$$

where†

$$\mathscr{H}(p, q) = \langle p, q|\hat{\mathscr{H}}|p, q\rangle. \tag{2.19}$$

Then the extremal solutions to (2.18) are given by

$$\dot{q} = \partial\mathscr{H}(p, q)/\partial p \qquad \dot{p} = -\partial\mathscr{H}(p, q)/\partial q \tag{2.20}$$

where $\mathscr{H}(p, q)$ in this case is analogous to the classical Hamiltonian (2.1). This method of obtaining a relationship between quantum and classical systems will be explored in our discussion of ultralocal scalar fields in the following sense.

† Whenever convenient we do not write explicitly the time dependence.

To any quantum system described by a Hamiltonian $\hat{\mathscr{H}}$, a pair of conjugate variables and some vacuum state, we can associate a classical system such as (2.20). The only problem that might arise is to check whether (2.19) really defines a classical Hamiltonian. This is not obvious when dealing with quantum field theory as we will do later. In fact a conjecture of Klauder called *weak correspondence principle* [19], WCP, says that given a quantum generator $\hat{\mathscr{G}}$ then the diagonal coherent state matrix elements of $\hat{\mathscr{G}}$ have the form of the classical generator (as $\hbar \to 0$). Consider for instance a neutral scalar field $\hat{\phi}(\boldsymbol{x})$ with canonical momentum $\hat{\pi}(\boldsymbol{x})$ in three-dimensional Euclidean space. Coherent states analogous to (2.9) are

$$|f, g\rangle = U[f, g]|0\rangle \tag{2.21}$$

$$U[f, g] = \exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^3x(f(\boldsymbol{x})\hat{\phi}(\boldsymbol{x}) - g(\boldsymbol{x})\hat{\Pi}(\boldsymbol{x}))\right) \tag{2.22}$$

where $f(\boldsymbol{x})$, $g(\boldsymbol{x})$ are infinitely differentiable smearing functions with compact support. Then one can show that the momentum, angular momentum and Hamiltonian operators have their classical counterparts as given by the WCP (we assume that these operators annihilate the vacuum state). For example the momentum operator $\hat{\mathscr{P}}_k = \int \mathrm{d}^3x\, \hat{\pi}(\boldsymbol{x})\nabla_k\hat{\phi}(\boldsymbol{x})$ gives

$$\mathscr{P}_k(f, g) \equiv \langle f, g|\hat{\mathscr{P}}_k|f, g\rangle = \int \mathrm{d}^3x f(\boldsymbol{x})\nabla_k g(\boldsymbol{x}) \tag{2.23}$$

which is the classical generator. Note that the smearing functions are playing the role of classical fields.

These considerations provide us with a framework for finding a formula analogous to (2.14) for ultralocal scalar fields in § 3. We will make an assumption concerning the form of the vacuum (cf (3.20), (3.27)) and then show explicitly that when the smearing functions evolve according to the RAP with the Hamiltonian as given by the WCP, the evolution of the wavefunction can be approximated by coherent states as $\hbar \to 0$.

3. Canonical and affine ultralocal fields

In this section we review the essential aspect of ultralocal quantum field theories. Classically the ultralocal scalar field is obtained by taking the Hamiltonian

$$\tilde{\mathscr{H}} = \int \mathrm{d}^3x[\tfrac{1}{2}\pi(x)^2 + (\nabla\phi(x))^2 + V(\phi(x))]$$

and dropping the $(\nabla\phi)^2$ term to obtain

$$\mathscr{H} = \int \mathrm{d}^3x[\tfrac{1}{2}\pi(x)^2 + V(\phi(x))]. \tag{3.1}$$

By dropping the spatial derivatives the evolution of the field at distinct spatial points is independent at all times. The light 'cone' at each $\boldsymbol{x} \in \mathbb{R}^3$ has collapsed to a timelike line passing through $\boldsymbol{x}$. The main point is that the quantisation of the Hamiltonian (3.1) can be accomplished exactly, without being forced to take $V(\phi)$ as a perturbation.

The quantum theory starts with the introduction of creation and annihilation operators $A^+(x, \lambda)$, $A(x, \lambda)$, $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^3$ acting on some Hilbert space H and a unit norm state $|0\rangle \in H$, called *vacuum state*, such that

$$[A(x, \lambda), A^+(x', \lambda')] = \delta(x - x')\delta(\lambda - \lambda') \tag{3.2}$$

$$[A, A] = 0 = [A^+, A^+] \tag{3.3}$$

$$A|0\rangle = 0. \tag{3.4}$$

In addition we define the operators

$$B(x, \lambda) = A(x, \lambda) + C(\lambda) \tag{3.5}$$

where $C(\lambda)$ is a real valued function satisfying $C(\lambda) = C(-\lambda)$ called *model function.* We assume that the vacuum state $|0\rangle$ is cyclic [20].

An overcomplete set of states in H is formed by

$$|\psi\rangle = \exp\left(\int\int d^3x\, d\lambda\, \psi(x, \lambda) A^+(x, \lambda)\right)|0\rangle \tag{3.6}$$

where $\psi \in h$, the Hilbert space of square integrable functions of x and λ, henceforth called *small Hilbert space.* Given the Hilbert space structure on h one infers a Hilbert space structure on H through the normalised inner product

$$\langle \psi' | \psi \rangle = \exp[-\tfrac{1}{2}\|\psi'\|^2 - \tfrac{1}{2}\|\psi\|^2 + (\psi', \psi)] \tag{3.7}$$

where (ψ, ϕ) is some inner product in h and $\| \ \|$ the associated norm.

We wish to have operators acting on H satisfying canonical commutation relations. For the ultralocal representations that we are studying this is not always possible. The operator representations are [1]

$$\hat{\phi}(x) = \int_{-\infty}^{+\infty} d\lambda\, B^+(x, \lambda)\lambda B(x, \lambda) \tag{3.8}$$

$$\hat{\pi}(x) = \int_{-\infty}^{+\infty} d\lambda\, B^+(x, \lambda)\frac{\hbar}{i}\frac{\partial}{\partial\lambda} B(x, \lambda) \tag{3.9}$$

satisfying

$$[\hat{\phi}(x), \hat{\pi}(x')] = i\hbar\delta(x - x') \int_{-\infty}^{+\infty} d\lambda\, B^+(x, \lambda) B(x', \lambda). \tag{3.10}$$

Up to the factor $\int d\lambda\, B^+B$ (which formally commutes with $\hat{\phi}$ and $\hat{\pi}$), (3.8) and (3.9) are canonically conjugate. For the most interesting representations (the irreducibles) one has $\int d\lambda\, C(\lambda)^2 = \infty$ and consequently $\int d\lambda\, B^+B$ and $\hat{\pi}(x)$ are not well defined [1]. The fact that the conjugate momentum is undefined makes using the WCP to obtain a classical limit for these theories more difficult. In fact we were unable to use the WCP in those cases where the model function is not square integrable and the discussion about canonical fields is concentrated on those models where the model function satisfy $\int d\lambda\, C(\lambda)^2 = M < \infty$. The affine momentum however (see below) is well defined even when C is not square integrable.

A consequence of the representation (8) of $\hat{\phi}$ is that the expectation functional

$$E(f) = \langle 0| \exp\left(i \int d^3x f(x)\hat{\phi}(x)\right)|0\rangle \tag{3.11}$$

where $f(x)$ is a differentiable function with compact support, satisfies the condition

$$E(f_1+f_2)=E(f_1)E(f_2) \tag{3.12}$$

for all f_1 and f_2 with disjoint supports. This statistical independence of disjoint spatial volumes is the essence of ultralocality. In addition, the truncated Green functions are all proportional to products of δ functions [1].

The representation of the Hamiltonian operator is given by

$$\hat{\mathscr{H}}(x)=\int \mathrm{d}\lambda\, B^{+}(x,\lambda)\,\hbar\!\!\!\!/\, B(x,\lambda) \tag{3.13a}$$

where

$$\hbar\!\!\!\!/ = -\tfrac{1}{2}\hbar^2\partial^2/\partial\lambda^2+v(\lambda). \tag{3.13b}$$

The potential $v(\lambda)$ is determined by the condition

$$\hat{\mathscr{H}}|0\rangle \equiv \int \mathrm{d}^3x\, \hat{\mathscr{H}}(x)|0\rangle = 0 \tag{3.14}$$

which implies

$$v(\lambda)=\hbar^2 C''(\lambda)/2C(\lambda). \tag{3.15}$$

Observe that the operator $\hbar\!\!\!\!/$ may also be written in the form

$$\hbar\!\!\!\!/ = a^{+}a \tag{3.16}$$

with

$$a=\frac{\hbar}{\sqrt{2}}C(\lambda)\frac{\partial}{\partial\lambda}C(\lambda)^{-1}.$$

The matrix elements of the Hamiltonian in the state $|\psi\rangle$ are particularly simple

$$\langle\psi|\hat{\mathscr{H}}(x)|\psi'\rangle=\langle\psi|\psi'\rangle\int \mathrm{d}\lambda\, \psi^*(x,\lambda)\,\hbar\!\!\!\!/\,\psi'(x,\lambda). \tag{3.17}$$

The operator $\hbar\!\!\!\!/$ acts only on the λ dependence of ψ and not on its x dependence. The time evolution operator also has a simple action on $|\psi\rangle$

$$\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{\mathscr{H}}t\right)|\psi\rangle=\left|\exp\left(-\frac{\mathrm{i}}{\hbar}\hbar\!\!\!\!/\,t\right)\psi\right\rangle. \tag{3.18}$$

From (3.18) it is clear that the dynamics in the field Hilbert space H reduces to that in the small Hilbert space h. In other words instead of doing quantum field theory in H we can equivalently do it in h. The simplicity of ultralocal field theory is to a large extent a consequence of this fact.

The above discussion points out the important role played by the function C. It determines the form of the potential and whether or not $\hat{\pi}$ is a well defined operator. If C is square integrable then $\hat{\pi}$ is well defined and the representation is reducible. In addition the ground state is not unique. If C is not square integrable and $\int \lambda^2(\lambda^2+1)^{-1}C^2(\lambda)\,\mathrm{d}\lambda=\infty$ then neither $\hat{\phi}$ nor $\hat{\pi}$ are well defined [1]. Thus we always assume

$$\int_{-\infty}^{+\infty}\mathrm{d}\lambda\, C^2\frac{\lambda^2}{\lambda^2+1}=N<\infty. \tag{3.19}$$

This condition also guarantees that (3.11) is well defined. An example of classes of square integrable and non-square-integrable model functions satisfying (3.19) are given respectively by

$$C(\lambda)=(|\lambda|)^{1/2}\exp\left(-\frac{1}{\hbar}y(\lambda)\right) \qquad C(\lambda)=(|\lambda|)^{-1/2}\exp\left(-\frac{1}{\hbar}y(\lambda)\right) \tag{3.20a, b}$$

where $y(\lambda)$ is an even polynomial. For concreteness one could take

$$y(\lambda)=\tfrac{1}{2}m\lambda^2. \tag{3.21}$$

With (3.21) the potential (3.15) corresponding to (3.20) becomes, respectively,

$$v(\lambda)=-\tfrac{1}{8}\hbar^2/\lambda^2+\tfrac{1}{2}m^2\lambda^2-m\hbar \qquad v(\lambda)=\tfrac{3}{8}\hbar^2/\lambda^2+\tfrac{1}{2}m^2\lambda^2. \tag{3.22a, b}$$

An alternative ultralocal theory that we will consider is based on the affine commutation relations [2]. The basic structure is as above except that we will replace λ with k which is restricted to $k>0$. We will represent operators $\hat{\phi}(x)$ and $\hat{K}(x)$ satisfying the affine commutation relations

$$[\hat{K}(x), \hat{\phi}(x')]=\mathrm{i}\hbar\delta(x-x')\hat{\phi}(x). \tag{3.23}$$

This is essentially the field version of the commutation relations of the Lie algebra of the affine group.

The affine ultralocal operators are

$$\hat{\phi}(x)=\int_0^\infty \mathrm{d}k\, B^+(x,k)kB(x,k) \tag{3.24}$$

$$\hat{K}(x)=\int_0^\infty \mathrm{d}k\, B^+(x,k)\frac{\hbar}{2\mathrm{i}}\left(\frac{\overleftarrow{\partial}}{\partial k}k-k\frac{\overrightarrow{\partial}}{\partial k}\right)B(x,k). \tag{3.25}$$

Observe that the spectrum of $\hat{\phi}(x)$ is positive since $k>0$.

The Hamiltonian for the affine case is taken to be

$$\hat{\mathscr{H}}(x)=\int \mathrm{d}k\, B^+(x,k)\,h\,B(x,k) \tag{3.26a}$$

with

$$h=-\hbar^2\frac{\partial}{\partial k}k\frac{\partial}{\partial k}+\frac{\hbar^2(\partial/\partial kk\,\partial/\partial kC)}{C} \tag{3.26b}$$

so that $\hat{\mathscr{H}}|0\rangle=0$. Following (3.20*b*) we take the model function to be

$$C(k)=k^{-1/2}\exp\left(-\frac{1}{\hbar}y(k)\right) \tag{3.27}$$

but in this case $y(k)$ does not need to be even. As in the canonical case there is a singularity in the potential at $k=0$ but this will cause no problem in applying the WCP and the RAP to obtain the quasiclassical approximation.

4. Quasiclassical approximation

A formula similar to (2.14) will be obtained for the affine field. Canonical commutation relations are treated later in this section.

Coherent states analogous to (2.9) are given by the following overcomplete set

$$|p, q\rangle = U[p, q]|0\rangle \tag{4.1}$$

$$U[p, q] = \exp\left(-\frac{i}{\hbar}\int d^3x\, q(x)\hat{\phi}(x)\right)\exp\left(\frac{i}{\hbar}\int d^3x \ln p(x)\hat{K}(x)\right) \tag{4.2}$$

with $p(x) > 0$. The real smearing functions p, q defined on $\mathbb{R}^3$ will be taken to be infinitely differentiable and of compact support. The unitary operators (4.2) expressed in terms of the affine fields $\hat{\phi}(x)$ and $\hat{K}(x)$ constitute a representation in the field Hilbert space H of the affine group since $U[p, q]U[p', q'] = U[pp', q + p^{-1}q']$ (the affine group on $\mathbb{R}$ is a two-parameter, non-Abelian group, given by $x \to a^{-1}x + b, \forall x \in \mathbb{R}$). Analogously when we work with canonical variables the unitary operators of the theory (see equation (4.36)) constitute a representation of the Heisenberg group in H.

The restricted action for the states (4.1) reads

$$\begin{aligned} I(p, q) &= \int dt\langle p, q|(i\hbar\partial/\partial t - \hat{\mathcal{H}})|p, q\rangle \\ &= \int dt\langle 0|\exp\left(-\frac{i}{\hbar}\int d^3x \ln p\hat{K}\right)\int d^3y(\dot{q}\hat{\phi} - (\dot{p}/p)\hat{K}) \\ &\quad \times \exp\left(\frac{i}{\hbar}\int d^3x \ln p\hat{K}\right)|0\rangle - \int dt\langle p, q|\hat{\mathcal{H}}|p, q\rangle \\ &= \int dt\, d^3x\, p\dot{q} - \int dt\langle p, q|\hat{\mathcal{H}}|p, q\rangle \end{aligned} \tag{4.3}$$

where $\hat{\mathcal{H}} = \int d^3x\, \hat{\mathcal{H}}(x)$ and $\hat{\mathcal{H}}(x)$ is given by (3.26). To obtain such a result we have imposed

$$\langle 0|\hat{\phi}(x)|0\rangle = 1 \tag{4.4}$$

and also we have used

$$U^{+}[p, q](\alpha\hat{\phi}(x) + \beta\hat{K}(x))U[p, q] = \alpha p(x)\hat{\phi}(x) + \beta(\hat{K}(x) + p(x)q(x)\hat{\phi}(x)). \tag{4.5}$$

Arbitrary variations with respect to p and q in (4.3) give the following 'classical' equations of motion for the smearing functions

$$\dot{q}_{cl}(x, t) = \frac{\delta\mathcal{H}(p_{cl}, q_{cl})}{\delta p_{cl}(x, t)} \qquad \dot{p}_{cl}(x, t) = -\frac{\delta\mathcal{H}(p_{cl}, q_{cl})}{\delta q_{cl}(x, t)} \tag{4.6a, b}$$

where we have defined

$$\mathcal{H}(p, q) = \langle p, q|\hat{\mathcal{H}}|p, q\rangle. \tag{4.7}$$

According to the RAP, (4.6) defines a suitable 'classical' orbit that can be used in quasiclassical approximations while, from the WCP, (4.7) is the corresponding 'classical' Hamiltonian. Thus we have all the necessary ingredients to approximate the evolution of a wavevector (3.18). Before doing this it is convenient to compute in more detail the form of (4.7).

Consider the derivative

$$\delta\mathcal{H}(p, q)/\delta q(x) = \frac{i}{\hbar}\langle p, q|[\hat{\phi}(x), \hat{\mathcal{H}}]|p, q\rangle = 2\langle p, q|\hat{K}(x)|p, q\rangle = 2p(x)q(x).$$

So

$$\mathscr{H}(p, q)=\int \mathrm{d}^3x\, pq^2+W(p) \tag{4.8}$$

where the 'potential' $W(p)$ is

$$\begin{aligned}
W(p)=\mathscr{H}(p,0) &= \langle 0|\exp\left(-\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^3x \ln p\hat{K}\right)\hat{\mathscr{H}}\exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^3x \ln p\hat{K}\right)|0\rangle \\
&= \langle 0|\left\{\hat{\mathscr{H}}+\left(-\frac{\mathrm{i}}{\hbar}\right)\int \mathrm{d}^3x \ln p[\hat{K},\hat{\mathscr{H}}]\right. \\
&\quad \left.+\frac{1}{2!}\left(-\frac{\mathrm{i}}{\hbar}\right)^2\iint \mathrm{d}^3x\,\mathrm{d}^3x' \ln p \ln p'[\hat{K},[\hat{K}',\hat{\mathscr{H}}]]+\ldots\right\}|0\rangle \\
&= \langle 0|\left\{\hat{\mathscr{H}}-\hbar^2\sum_{n=1}^{\infty}\iint \mathrm{d}^3x\,\mathrm{d}k\frac{(-\ln p)^n}{n!}B^{+}\vec{\partial}k\,\vec{\partial}B\right. \\
&\quad \left.+\hbar^2\sum_{n=1}^{\infty}\iint \mathrm{d}^3x\,\mathrm{d}k\frac{(\ln p(k\vec{\partial}))^n}{n!}\left(\frac{\vec{\partial}k\,\vec{\partial}C}{C}\right)B\right\}|0\rangle \\
&= \iint \mathrm{d}^3x\,\mathrm{d}k\langle 0|B^{+}(x,k)\hbar(kp)B(x,k)|0\rangle \\
&= \iint \mathrm{d}^3x\,\mathrm{d}k\, C(k)\hbar(kp)C(k)
\end{aligned} \tag{4.9}$$

and use has been made of $\mathrm{e}^{gk\vec{\partial}}f(k)=f(k\mathrm{e}^{g})$ for differentiable f, g with $\vec{\partial}\equiv\partial/\partial k$. It can be shown that, writing the operator $\hbar$ as

$$\hbar=a^{+}a \tag{4.10}$$

for $a=\hbar\sqrt{k}C(k)(\partial/\partial k)C(k)^{-1}$, (4.9) becomes

$$\begin{aligned}
W(p) &= \iint \mathrm{d}^3x\,\mathrm{d}k\, p^{-1}|\sqrt{k}aC(kp^{-1})|^2 \\
&= \hbar^2\iint \mathrm{d}^3x\,\mathrm{d}k\, kC^2(k)p^{-1}\left|\frac{\partial}{\partial k}\frac{C(kp^{-1})}{C(k)}\right|^2.
\end{aligned} \tag{4.11}$$

The evolution (3.18) can be expressed in the small Hilbert space h as

$$\exp[-(\mathrm{i}/\hbar)\hbar t]\psi(x,k)=\psi(x,k,t). \tag{4.12}$$

Coherent states in the small Hilbert space that approximate the RHS of (4.12) are obtained as follows. Since the states (3.6) are eigenstates of A,

$$A(x,k)|\psi\rangle=\psi(x,k)|\psi\rangle \tag{4.13}$$

the vector $|\psi_{p,q}\rangle$ of the Hilbert space H corresponding to a coherent state $|p,q\rangle\in H$ is determined by the following element of h:

$$\psi_{p,q}(x,k)=\exp[-(\mathrm{i}/\hbar)q(x)k]p^{-1/2}(x)C(kp^{-1}(x))-C(k) \tag{4.14}$$

(just compute $A(x,k)|p,q\rangle$ to obtain $\psi_{p,q}(x,k)|p,q\rangle$).

The Schrödinger equation in h reads

$$\hbar\psi(k,t)=\mathrm{i}\hbar\dot{\psi}(k,t) \tag{4.15}$$

where it is not necessary to write explicitly the x dependence since, due to ultralocality, the dynamics at the spatial point x_1 is a copy of that at x_2, for any x_1, x_2. Thus our problem has been reduced to 'quantum mechanics in k space'. When some initial condition is chosen, (4.15) is equivalent to (4.12). We will always assume that the initial state is a coherent state

$$\psi_0(k) \equiv \psi_{p_0,q_0}(k) \tag{4.16}$$

with $p_0 \equiv p(0)$, $q_0 \equiv q(0)$. Observe that if we were dealing with an exact system then (4.16) would evolve to another coherent state and we could have written (4.15) as

$$\not{h}\psi_{p,q}(k) = \mathrm{i}\hbar\dot{\psi}_{p,q}(k) \tag{4.17}$$

where the t dependence is entirely contained in p and q. For instance this kind of evolution occurs when the model function is constant. In general we will show that when $C(k)$ has the form (3.27) then system (4.6) with Hamiltonian (4.8)

$$\dot{q}_{\mathrm{cl}} = q_{\mathrm{cl}}^2 + \delta W(p_{\mathrm{cl}})/\delta p_{\mathrm{cl}} \tag{4.18a}$$

$$\dot{p}_{\mathrm{cl}} = -2p_{\mathrm{cl}}q_{\mathrm{cl}} \tag{4.18b}$$

give, as $\hbar \to 0$, an approximate solution to the Schrödinger equation when we choose a suitable norm in the small Hilbert space. This is the main result of the paper and we state it as a theorem.

Theorem. Suppose we have a set $p_{\mathrm{cl}}(t)$, $q_{\mathrm{cl}}(t)$ of smearing functions whose evolution (4.18) defines a 'classical' orbit. Then coherent states (4.14) defined on this orbit satisfy

$$\lim_{\hbar\to 0} \|\not{h}\psi_{p_{\mathrm{cl}}(t),q_{\mathrm{cl}}(t)} - \mathrm{i}\hbar\dot{\psi}_{p_{\mathrm{cl}}(t),q_{\mathrm{cl}}(t)}\| = 0. \tag{4.19}$$

Sometimes one writes (4.19) loosely as

$$\not{h}\psi_{p_{\mathrm{cl}}(t),q_{\mathrm{cl}}(t)} \overset{\hbar\to 0}{\approx} \mathrm{i}\hbar\dot{\psi}_{p_{\mathrm{cl}}(t),q_{\mathrm{cl}}(t)}$$

or, equivalently, one says that the right-hand side of (4.12) is approximated by a coherent state defined on the 'classical' orbit

$$\exp[(-\mathrm{i}/\hbar)\not{h}t]\psi_{p_{\mathrm{cl}}(0),q_{\mathrm{cl}}(0)} \overset{\hbar\to 0}{\sim} \psi_{p_{\mathrm{cl}}(t),q_{\mathrm{cl}}(t)}. \tag{4.20}$$

In order to prove the theorem we must first state what kind of norm should be used in the limit (4.19). The measure for this system† is obtained from $\mathrm{d}\mu = \mathrm{d}k\, k^{-1}C^2(k)$ by a change of variable: $\mathrm{d}\mu_p = \mathrm{d}k\, k^{-1}C^2(kp^{-1})$ (from (4.4) $\int_0^{\infty}\mathrm{d}\mu\, k^2 = 1$). In what follows we use $\mathrm{d}\mu_p$ and the convergence in (4.19) is given in terms of the norm $\|\varphi\|^2 = \int \mathrm{d}\mu_p\, \varphi^*\varphi$. Also the subscript 'cl' and the time dependence in $p_{\mathrm{cl}}(t)$, $q_{\mathrm{cl}}(t)$ will not be written.

† Among the measures of the form $\mathrm{d}\nu = k^{-n}C^2(k)$, the only one that will give sensible results in the context of affine fields is the singular measure with $n = 1$. Anything more singular ($n > 1$) will imply that none of the integrals in (4.26) is well defined while anything less singular will lead to the conclusion that (4.19) is always true whether (4.18) is satisfied or not (to see this just use the same arguments that led to (4.28) and conclude that (4.26′) is always $\mathrm{o}(\hbar)$).

In the case of canonical models any normalised measure $\mathrm{d}\nu = N\lambda^n C^2(\lambda)\,\mathrm{d}\lambda$, $n \geq 0$, can be used where N is some normalisation constant. However singular measures will imply that (4.41) is not well defined.

It is necessary now to express (4.18) in the small Hilbert space:

$$\int_0^\infty \mathrm{d}\mu\, k^2(\dot q - q^2) = \int_0^\infty \mathrm{d}\mu\, k^2 \bar w \tag{4.21}$$

where $\bar w$ is a representation of $\delta W(p)/\delta p$ in h. From (4.9),

$$\begin{aligned}\frac{\delta W(p)}{\delta p} &= \int_0^\infty \mathrm{d}\mu\, k^2(y'^2(kp) - y'^2(k)p^{-2}) + 2\int_0^\infty \mathrm{d}\mu\, k^2(k(y'(kp)p - y'(k))y''(kp))\\ &= \int_0^\infty \mathrm{d}\mu_p\, k^2 p^{-2}(y'^2 - y_p'^2 p^{-2}) + \int_0^\infty \mathrm{d}\mu_p\, k^2 p^{-2}(2kp^{-1}(y'p - y_p')y'')\\ &\equiv \int \mathrm{d}\mu_p\, k^2 \bar w_p\end{aligned}$$

and we obtain

$$\bar w_p = p^{-2} w_p \qquad w_p = y'^2 - y_p'^2 p^{-2} + 2kp^{-1}(y'p - y_p')y'' \tag{4.22}$$

where $y \equiv y(k)$, $y_p \equiv y(kp^{-1})$ and y', y_p' is the derivative with respect to k and kp^{-1} respectively.

Thus (4.18) is equivalent to

$$\dot q - q^2 = w_p \qquad \dot p = -2pq. \tag{4.23a, b}$$

Using the Hamiltonian (3.26b) and the model (3.27) we have (set $\psi_{p,q}^C = \psi_{p,q} + C$):

$$\begin{aligned}\mathscr{h}\psi_{p,q} &= (-\hbar^2\vec\partial k\vec\partial + \hbar^2 C^{-1}(\vec\partial k\vec\partial C))\psi_{p,q}^C\\ &= -\hbar^2(\vec\partial + k\vec\partial^2)\psi_{p,q}^C + \hbar^2\left(\frac{1}{4k} + \frac{1}{\hbar^2}ky'^2 - \frac{k}{\hbar}y''\right)\psi_{p,q}^C\\ &= -\hbar^2\Big[(-(\mathrm{i}/\hbar)q + C_p' p^{-1}C_p^{-1}) + k[(C_p'' p^{-2}C_p^{-1} - C_p'^2 p^{-2}C_p^{-2})\\ &\quad + (-(\mathrm{i}/\hbar)q + C_p' p^{-1}C_p^{-1})^2] - \left(\frac{1}{4k} + \frac{1}{\hbar^2}ky'^2 - \frac{k}{\hbar}y''\right)\Big]\psi_{p,q}^C\\ &= [\hbar k(y_p'' p^{-2} - y'') + (kq^2 - 2\mathrm{i}qkp^{-1}y_p' + ky'^2 - ky_p'^2 p^{-2})]\psi_{p,q}^C \qquad (4.24)\\ \mathrm{i}\hbar\dot\psi_{p,q} &= \mathrm{i}\hbar[-(\mathrm{i}/\hbar)\dot q k - \tfrac12 p^{-1}\dot p - C_p' p^{-2}\dot p k C_p^{-1}]\psi_{p,q}^C\\ &= (\dot q k + \mathrm{i}y_p' p^{-2}\dot p k)\psi_{p,q}^C \qquad (4.25)\end{aligned}$$

where $C_p \equiv C(kp^{-1})$. Thus

$$\begin{aligned}\|\mathscr{h}\psi_{p,q} - \mathrm{i}\hbar\dot\psi_{p,q}\|^2 &= \int_0^\infty \mathrm{d}\mu_p\{k^2[\hbar(y_p'' p^{-2} - y'') + (\dot q - q^2 + y'^2 - y_p' p^{-2})]^2\\ &\quad + k^2[(2pq + \dot p)y_p' p^{-2}]^2\}. \qquad (4.26)\end{aligned}$$

From (4.23b) $2pq + \dot p = 0$ and, defining

$$\mathrm{o}(\hbar) = \hbar A \equiv 2\hbar \int_0^\infty \mathrm{d}\mu_p\, k^2(y_p'' p^{-2} - y'')(q^2 - \dot q + y'^2 - y_p'^2 p)$$

$$\mathrm{o}(\hbar^2) = \hbar^2 B \equiv \hbar^2 \int_0^\infty \mathrm{d}\mu_p\, k^2(y_p'' p^{-2} - y'')^2$$

with A and B well defined integrals, (4.26) becomes

$$\|\hbar\psi_{p,q} - \mathrm{i}\hbar\dot{\psi}_{p,q}\|^2 = \int_0^\infty \mathrm{d}\mu_p\, k^2(q^2 - \dot{q} + y'^2 - y_p'^2 p^{-2})^2 + \mathrm{o}(\hbar) + \mathrm{o}(\hbar^2). \tag{4.26'}$$

From (4.22), (4.23a),

$$\|\hbar\psi_{p,q} - \mathrm{i}\hbar\dot{\psi}_{p,q}\|^2 = \int_0^\infty \mathrm{d}\mu_p\, k^2[2kp^{-1}(y'p - y_p')y'']^2 + \mathrm{o}(\hbar) + \mathrm{o}(\hbar^2). \tag{4.27}$$

The integral on the RHS of (4.27) is of order $\hbar^2$:

$$\begin{aligned}\int_0^\infty \mathrm{d}\mu_p\, k^2[2k(y'p - y_p')p^{-1}y'']^2 &= \int_0^\infty \mathrm{d}\mu\, p^2k^2[2k(y'(kp)p - y')y''(kp)]^2\\ &= \int_0^\infty \mathrm{d}k\, p^2 N(\hbar)\exp(-(1/\hbar)y(k))[2k(y'(kp)p - y')y''(kp)]^2\\ &= \hbar^2\int_0^\infty \mathrm{d}k\, p^2 N(\hbar)\hbar\exp(-uz(\hbar u))[2u(y'(\hbar up)p - y'(\hbar u))y''(\hbar up)]^2\end{aligned} \tag{4.28}$$

where we defined a change of variable by setting $\hbar^{-1}y(k) \equiv \hbar^{-1}kz(k) \equiv uz(\hbar u)$ with $\hbar u = k$; $N(\hbar)$ is a normalisation constant (from $\int \mathrm{d}\mu\, k^2 = 1$ we conclude that $N(\hbar)\hbar\exp(-uz(\hbar u))$ is of order zero and (28) is indeed $\mathrm{o}(\hbar^2)$). Thus

$$\|\hbar\psi_{p,q} - \mathrm{i}\hbar\dot{\psi}_{p,q}\| = \mathrm{o}(\sqrt{\hbar}) \tag{4.29}$$

and the theorem follows.

These methods can be extended to the case of canonical models. However, as we will see, only square integrable model functions give sensible results; non-square integrable models seem to be incompatible with the WCP and RAP.

Theorem. Suppose we have a set p, q of smearing functions evolving according to

$$\dot{q} = \delta\mathcal{H}(p, q)/\delta p \qquad \dot{p} = -\delta\mathcal{H}(p, q)/\delta q \tag{4.30a, b}$$

with

$$\mathcal{H}(p, q) = \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}\lambda\, C^2(\lambda)\int \mathrm{d}^3x\, p^2 + W(q) \tag{4.31}$$

$$W(q) = \iint \mathrm{d}^3x\, \mathrm{d}\lambda\, C(\lambda - q)\hbar C(\lambda - q). \tag{4.32}$$

Then

$$\lim_{\hbar\to 0}\|\hbar\psi_{p,q} - \mathrm{i}\hbar\dot{\psi}_{p,q}\| = 0 \tag{4.33}$$

where $\hbar$ is given by (3.13b) and

$$\psi_{p,q} = \alpha\exp[-(\mathrm{i}/\hbar)p\lambda]C(\lambda + q) - C(\lambda) \tag{4.34}$$

is analogous to (4.14) (a phase $\alpha = \exp[-(\mathrm{i}/2\hbar)\int_0^t \mathrm{d}s\, p^2(s)]$ has been included for later convenience); the model function is (3.20a).

The Hamiltonian (4.31) is obtained from the WCP while the equations of motion (4.30) are obtained from the RAP. It is clear that when $C(\lambda)$ is not square integrable (4.30), (4.31) will not make sense (only when C is a constant can we 'regularise' these integrals and still obtain (4.30) from the WCP and the RAP; this very special choice was made in [4]). The coherent state (4.34) in the small Hilbert space is obtained from

$$|p, q\rangle = U[p, q]|0\rangle \tag{4.35}$$

$$U[p, q] = \exp\left(-\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^3x\, p(x)\hat{\phi}(x)\right)\exp\left(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^3x\, q(x)\hat{\pi}(x)\right) \tag{4.36}$$

by using the same arguments that led to (4.14).

The theorem is proved by first rewriting (4.30) in the small Hilbert space:

$$\dot{q} = p \qquad \dot{p} = w_q \tag{4.37a, b}$$

where w_q is a representation of $\delta W(q)/\delta q$ in h. From (4.32), (3.15), (3.13b),

$$W(q) = \int\int \mathrm{d}^3x\, \mathrm{d}\lambda\, C^2(\lambda - q)\frac{1}{2}\left[\hbar^2\left(\frac{1}{4(\lambda-q)^2} - \frac{1}{4\lambda^2}\right)\right.$$
$$\left. + \hbar\left(y''(\lambda - q) - y'' + \frac{1}{\lambda - q}y'(\lambda - q) - \frac{1}{\lambda}y'\right) - [y'^2(\lambda - q) - y'^2]\right]$$

$$-\frac{\delta W(q)}{\delta q} = \int_{-\infty}^{+\infty} \mathrm{d}\lambda\, C^2(\lambda - q)\left\{\left[\hbar^2\left(\frac{1}{4(\lambda - q)^2} - \frac{1}{4\lambda^2}\right)\right.\right.$$
$$\left. + \hbar\left(y''(\lambda - q) - y'' + \frac{1}{\lambda - q}y'(\lambda - q) - \frac{1}{\lambda}y'\right) - [y'^2(\lambda - q) - y'^2]\right]$$
$$\left.\times\left(\frac{1}{2(\lambda - q)} - \frac{1}{\hbar}y'(\lambda - q)\right)\right\}$$

where

$$C'(\lambda) = C(\lambda)\left(\frac{1}{2\lambda} - \frac{1}{\hbar}y'\right) \qquad C''(\lambda) = C(\lambda)\left(-\frac{1}{4\lambda^2} - \frac{1}{\hbar}y'' - \frac{1}{\hbar\lambda}y' + \frac{1}{\hbar^2}y'^2\right)$$
$$y \equiv y(\lambda)$$

and some odd integrands have been dropped. Changing variables, $\lambda \to \lambda + q$,

$$\frac{\delta W(q)}{\delta q} = \int_{-\infty}^{+\infty} \mathrm{d}\lambda\, C^2(\lambda)\left\{\left[\hbar^2\left(\frac{1}{4(\lambda + q)^2} - \frac{1}{4\lambda^2}\right)\right.\right.$$
$$\left.\left. + \hbar\left(y_q'' - y'' + \frac{1}{\lambda + q}y_q' - \frac{1}{\lambda}y'\right) - (y_q'^2 - y'^2)\right]\left(\frac{1}{2\lambda} - \frac{1}{\hbar}y'\right)\right\}$$

where $y_q \equiv y(\lambda + q)$ and y', y_q' mean derivative with respect to λ and $\lambda + q$ respectively. Thus

$$w_q = \left[\hbar^2\left(\frac{1}{4(\lambda + q)^2} - \frac{1}{4\lambda^2}\right) + \hbar\left(y_q'' - y'' + \frac{1}{\lambda + q}y_q' - \frac{1}{\lambda}y'\right) + (y'^2 - y_q'^2)\right]\left(\frac{1}{2\lambda} - \frac{1}{\hbar}y'\right). \tag{4.38}$$

Using (4.34), (3.13b), (with $\psi_{p,q}^C = \psi_{p,q} + C$)

$$\mathscr{h}\psi_{p,q}=(-\tfrac{1}{2}\hbar^2\vec{\partial}+\tfrac{1}{2}\hbar^2C^{-1}C'')\psi^C_{p,q}$$

$$=\left\{-\frac{\hbar^2}{2}\left[(C''_qC_q^{-1}-C'^2_qC_q^{-2})+\left(-\frac{\mathrm{i}}{\hbar}p+C'_qC_q^{-1}\right)^2\right]\right.$$

$$\left.+\frac{\hbar^2}{2}\left(-\frac{1}{4\lambda}-\frac{1}{\hbar}y''-\frac{1}{\hbar\lambda}y'+\frac{1}{\hbar^2}y'^2\right)\right\}\psi^C_{p,q}$$

$$=\left[\frac{\hbar^2}{2}\left(\frac{1}{4(\lambda+q)^2}-\frac{1}{4\lambda^2}\right)+\frac{\hbar}{2}\left(y''_q-y''+\frac{1}{\lambda+q}y'_q-\frac{1}{\lambda}y'_q+\frac{\mathrm{i}p}{\lambda+q}\right)\right.$$

$$\left.+\frac{1}{2}(p^2-2\mathrm{i}py'_q+y'^2-y'^2_q)\right]\psi^C_{p,q} \tag{4.39}$$

$$\mathrm{i}\hbar\dot{\psi}_{p,q}=\mathrm{i}\hbar\left(-\frac{\mathrm{i}}{2\hbar}p^2-\frac{\mathrm{i}}{\hbar}\dot{p}\lambda+C'_q\dot{q}C_q^{-1}\right)\psi^C_{p,q}$$

$$=\mathrm{i}\hbar\left\{-\frac{\mathrm{i}}{2\hbar}p^2-\frac{\mathrm{i}}{\hbar}\dot{p}\lambda+\dot{q}\left(\frac{1}{2(\lambda+q)}-\frac{1}{\hbar}y'_q\right)\right\}\psi^C_{p,q}$$

$$=\left[\frac{1}{2}p^2+\dot{p}\lambda+\mathrm{i}\dot{q}\left(\frac{\hbar}{2(\lambda+q)}-y'_q\right)\right]\psi^C_{p,q} \tag{4.40}$$

(note that, due to the phase $\exp(-(\mathrm{i}/\hbar)\int_0^t \mathrm{d}s\,p^2(s))$, there is a term $\frac{1}{2}p^2$ in (4.40)). Assuming from now on that $\mathrm{d}\mu=C^2\,\mathrm{d}\lambda$ is normalised we study the convergence (4.33) in terms of $\|\varphi\|^2=\int\mathrm{d}\mu\,\varphi^*\varphi$

$$\|\mathscr{h}\psi_{p,q}-\mathrm{i}\hbar\dot{\psi}_{p,q}\|^2$$

$$=\int_{-\infty}^{+\infty}\mathrm{d}\mu\left\{\left[\left(\frac{\hbar}{2(\lambda+q)}-y'_q\right)(p-\dot{q})\right]^2+\left[\frac{\hbar^2}{2}\left(\frac{1}{4(\lambda+q)^2}-\frac{1}{4\lambda^2}\right)\right.\right.$$

$$\left.\left.+\frac{\hbar^2}{2}\left(y''_q-y''+\frac{1}{\lambda+q}y'_q-\frac{1}{\lambda}y'\right)+\frac{1}{2}(y'^2-y'^2_q)-\dot{p}\lambda\right]^2\right\}.$$

Using (4.38), (4.37),

$$\|\mathscr{h}\psi_{p,q}-\mathrm{i}\hbar\dot{\psi}_{p,q}\|^2$$

$$=\int_{-\infty}^{+\infty}\mathrm{d}\mu\left\{\left[\hbar^2\left(\frac{1}{4(\lambda+q)^2}-\frac{1}{4\lambda^2}\right)\right.\right.$$

$$\left.\left.+\hbar\left(y''_q-y''+\frac{1}{\lambda+q}y'_q-\frac{1}{\lambda}y''\right)-(y'^2-y'^2_q)\right]\frac{\lambda}{\hbar}y'\right\}^2. \tag{4.41}$$

When squaring the integrand in (4.41) we can see that all terms are of order at least $\hbar^2$ (use the fact that $y(\lambda)$ is an even polynomial, so $y'(\lambda)=\Sigma\, a_{2n}\lambda^{2n-1}$, $n>0$, and, under a change of variable similar to that in (4.28), $y'(\hbar u)^{2n-1}=\Sigma\, a_{2n}(\hbar u)^{2n-1}$ is $\mathrm{o}(\hbar)$). Thus we conclude

$$\|\mathscr{h}\psi_{p,q}-\mathrm{i}\hbar\dot{\psi}_{p,q}\|=\mathrm{o}(\hbar).$$

Thus the WCP and the RAP define a suitable 'classical' orbit upon which coherent states can be defined that will approximate, in the sense of norms in the small Hilbert space, the quantum evolution of affine and canonical (square integrable) ultralocal fields for small $\hbar$.

5. Discussion and conclusions

The simplicity of ultralocal field theory stems from the fact that we can work in the small Hilbert space with finite degrees of freedom instead of the field Hilbert space. We have seen that affine fields are less singular than canonical fields and that the approximation scheme in the small Hilbert space studied so far breaks down when non-square-integrable model functions are used in the canonical context. This is because we were unable to get rid of an infinity of the form $\int d\lambda\, C(\lambda)^2$ which completely hinders the use of the WCP and the RAP to define the 'classical' orbit.

The ideas presented in this paper have already been applied to some cosmological situations where the gravitational field exhibits a kind of spontaneously decoupled dynamics in the asymptotic region close to the initial singularity (the gravitational field is ultralocal in that region for a large class of solutions to Einstein field equations [21-23]. Strong coupling Yang-Mills fields can also be treated as an ultralocal field and this topic will be tackled in a future publication.

Although we recognise that the concept of ultralocality must be weakened in order that operators like $(\nabla\phi)^2$ be represented, the essential ideas proved to be a useful guideline for future developments in those theoretical contexts where ultralocality may be relevant.

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